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ON COALITION FORMATION IN SIMPLE GAMES: A MATHEMATICAL ANALYSIS--ETC(U)
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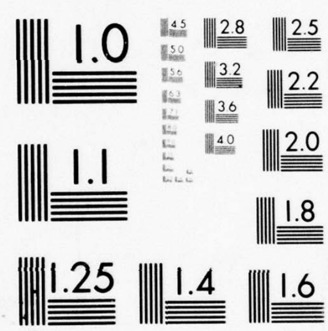

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ON COALITION FORMATION IN SIMPLE GAMES: A MATHEMATICAL ANALYSIS OF CAPLOW'S AND GAMSON'S THEORIES

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ABSTRACT

In this paper, we propose a theory of coalition formation in simple games. The process of coalition formation is modeled as an abstract game. Two solutions of abstract games, the core and the dynamic solution, are used as the predictions of our model. Two classical theories of coalitions in sociology due to Caplow and Gamson are reformulated in a more general and mathematical setting. These theories are then analyzed using the techniques of our theory.

AMS(MOS) Subject Classification: 90D12, 90D45

Key Words: coalition formation, simple games, Caplow's theory, Gamson's theory, core, dynamic solution, abstract games

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SIGNIFICANCE AND EXPLANATION

A simple game is a pair $(N,\mathcal{W})$ where N is a set of players and $\mathcal{W}$ is a set of all winning coalitions. (A coalition is a subset of N). Given a simple game, one is interested in predicting the coalition that will actually form.

Simple games are mathematical abstractions of decision making institutions in real life such as legislatures, committees, elections etc. Coalition formation in simple games has been the subject of numerous empirical and theoretical studies in the social sciences. However, most of these theories are of a rather ad hoc nature.

In this paper, we propose a theory based on the theory of n-person cooperative games. Two classical, non-mathematical theories of coalition formation in sociology due to Caplow and Gamson are reformulated in a more general and mathematical setting, and analyzed using the techniques of our theory.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

ON COALITION FORMATION IN SIMPLE GAMES:
A MATHEMATICAL ANALYSIS OF CAPLOW'S AND GAMSON'S THEORIES

Prakash P. Shenoy

1. Introduction

This paper deals with the question of coalition formation in simple games. Coalition formation has been the subject of many empirical and theoretical studies in the social sciences. There are a number of simple theories which essentially consist of a hypothesis concerning the player's goals or motives, a premise concerning their payoffs and an inference which singles out the coalitions most likely to form. Some of these theories are reviewed in Shenoy [26].

Regarding simple games, the main thrust of the research in game theory has been in determining an index which measures the power of each player. Here, we model the process of coalition formation as an abstract game. The core and the dynamic solution of the abstract game are then used as the predictions of our model.

Two classical theories of coalition formation due to Caplow and Gamson are reformulated in a slightly more general and mathematical setting. These theories are then analyzed using the techniques of our theory.

In Section 2, we review the core and the dynamic solution of abstract games. Simple games are introduced in Section 3. Our model of coalition formation is presented in Section 4. Section 5 contains a representation of our model by means of directed graphs. The predictions of our model are then described in graph theoretic terminology. The mathematical analysis of Caplow's and Gamson's theories are presented in Section 6 and 7 respectively. Finally, Section 8 contains some concluding remarks.

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024 and the National Science Foundation under Grant No. MCS75-17385 A01.

2. The core and the dynamic solution of abstract games.

An abstract game is a pair (X, dom) where X is an arbitrary set whose members are called outcomes of the game, and dom is an arbitrary binary relation defined on X and is called domination. An outcome $x \in X$ is said to be accessible from an outcome $y \in X$, denoted by $x \leftarrow y$ (or $y \rightarrow x$), if there exists outcomes $z_0 = x, z_1, z_2, \ldots, z_{m-1}, z_m = y$ where m is a positive integer such that

$$x = z_0 \text{ dom } z_1 \text{ dom } z_2 \text{ dom } \ldots \text{ dom } z_{m-1} \text{ dom } z_m = y. \tag{2.1}$$

Also assume $x \leftarrow x$, i.e. an outcome is accessible from itself. Clearly the binary relation accessible is transitive.

An interpretation of the relation accessible is as follows: If the players are considering an outcome y at some stage, then an outcome they will consider next will be a $z \in X$ such that $z \text{ dom } y$. If $x \leftarrow y$ and if the players are considering outcome y at some time, then it is possible that they will consider outcome x at some future time. I.e. one may interpret the relation as a possible succession of transitions from one outcome to another.

Two outcomes x and y which are accessible to each other are said to communicate and we write this as $x \leftrightarrow y$. Since the relation accessible is transitive and reflexive it follows that communication is an equivalence relation. We can now partition the set X into equivalence classes. Two outcomes are in the same equivalence class if they communicate with each other.

The core C (due to Gillies [14] and Shapley) of an abstract game is defined to be the set of undominated outcomes. We can rewrite the definition of the core in terms of the relation accessible as follows

$$C = \{x \in X : \text{For all } y \in X, y \neq x, \text{ we have } y \not\leftarrow x\}. \tag{2.2}$$

I.e., in the terminology of Markov chains, the core is the set of all absorbing outcomes. Note that each outcome in the core (if nonempty) is an equivalence class by itself.

We define an elementary dynamic solution (elem. d-solution) of an abstract game (X,dom) as a set $S \subset X$ such that

$$\text{if } x \in S,\ y \in X-S, \text{ then } y \not\leftarrow x \text{ and} \tag{2.3}$$

$$\text{if } x,y \in S, \text{ then } y \leftarrow x \text{ and } x \leftarrow y. \tag{2.4}$$

Condition (2.3) requires S to be 'externally stable' in a dynamic sense, i.e., if the players are considering $x \in S$ at some time, then they will never consider any outcome that is not in

S in the future. We can think of Condition (2.4) as 'internal stability' in a dynamic sense. I.e., it the players make a transition (in the consideration of outcomes) from x to y, then it is possible that the players will again reconsider the outcome x in the future.

Note that an elem. d-solution is an equivalence class. The converse, however, is not always true, i.e., an equivalence class need not be an elem. d-solution. Condition (2.3) requires S to be (in the terminology of Markov chains) a non-transient equivalence class. Also note that each outcome in the core is an elem. d-solution.

The dynamic solution (d-solution) P of an abstract game is the union of all distinct elementary dynamic solutions. I.e.,

(2.5) $\quad P = \cup \{S \subset X : S \text{ is an elem. d-solution}\}$.

The following are easy consequences of the definition.

Proposition 2.1. Let $\Gamma = (X, \text{dom})$ be any abstract game. Then $C \subset P$.

Theorem 2.2. If X is a finite set, then the dynamic solution of the abstract game (X, dom) is always nonempty and is a unique set.

Proof. See Shenoy [27].

The dynamic solution has also been defined independently by Kalai, Pazner and Schmeidler [17, 18].

3. Simple Games.

Let $N = \{1,\dots,n\}$ denote the set of players. Nonempty subsets of N are called coalitions. A simple game can be represented by a pair $(N,\mathcal{W})$ where $\mathcal{W}$ is the set of all winning coalitions. A simple game is monotonic iff $R \in \mathcal{W}$, $T \supset R \Rightarrow T \in \mathcal{W}$, and proper iff $R \in \mathcal{W} \Rightarrow N-R \notin \mathcal{W}$. Proper simple games are always monotonic. A winning coalition R is called minimal winning if every proper subset of R is non-winning. A monotonic simple game can be represented by the pair $(N,\mathcal{W}^m)$ where $\mathcal{W}^m$ is the set of all minimal winning coalitions. If $\mathcal{W}^m = \{\{i\}\}$, then player i is said to be a dictator. If $j \in \cap\mathcal{W}^m \neq \phi$, then player j is said to be a veto player. If $k \notin \cup\mathcal{W}^m$ then player k is said to be a dummy. A weighted majority game is a monotonic simple game that can be represented by

$$[q \, ; \, a_1,\dots,a_n] \tag{3.1}$$

where $q \geq 0$ is called the quota, $a_i \geq 0$ is the weight of the i^{th} player and $R \in \mathcal{W} \Leftrightarrow \sum_{i \in R} a_i \geq q$. Expression (3.1) is said to be a weighted majority representation of the simple game. Note that the weighted majority game represented by (3.1) is proper if $q > (a_1 + a_2 + \dots + a_n)/2$.

4. A Model of Coalition Formation.

Let Γ be a n-person simple game. Let 2^N denote the set of all nonempty subsets (coalitions) of N and Π denote the set of all partitions (coalition structures) of N. Let $\mathcal{S} : \Pi \to E^n$ be a power index (p.i.) where E^n denotes the n-dimensional Euclidean space. Intuitively, given that players in N align themselves into coalitions in the coalition structure (c.s.) $\rho \in \Pi$, we interpret $\mathcal{S}(\rho)$ as a vector in E^n whose i^{th} component $\mathcal{S}(\rho)(i)$ is a numerical measure of player i's power. E.g. $\mathcal{S}$ may denote the Shapley-Shubik power index, the Banzhaf-Coleman power index, the nucleolus, etc.

We can regard Π as the set of outcomes of an abstract game. We define a binary relation on Π as follows.

Let $\rho_1, \rho_2 \in \Pi$, and $\mathcal{S}$ be a p.i. Then ρ_1 dominates ρ_2 with respect to p.i. $\mathcal{S}$, denoted by $\rho_1 \text{ dom}(\mathcal{S}) \rho_2$, iff

$$\exists \text{ a nonempty } R \in \rho_1 \text{ such that } \mathcal{S}(\rho_1)(i) > \mathcal{S}(\rho_2)(i) \quad \forall\, i \in R$$

Intuitively, if $\rho_1 \text{ dom}(\mathcal{S}) \rho_2$, then the players in some coalition R in c.s. ρ_1 prefer ρ_1 to ρ_2. We require the players in subset R to be together in a coalition in c.s. ρ_1 so that there is no conflict of interest between these players' preference for ρ_1 and their allegiance to the other players in their coalition.

The dominance relation as defined above may be neither asymmetric nor transitive. We now have an abstract game $(\Pi, \text{dom}(\mathcal{S}))$ where Π is the set of outcomes and $\text{dom}(\mathcal{S})$ is a binary relation on Π. Let $K_0(\mathcal{S})$ and $K_1(\mathcal{S})$ denote the core and the dynamic solution respectively of this abstract game. By Proposition (2.1), we have $K_0(\mathcal{S}) \subset K_1(\mathcal{S})$. It is conceivable that $K_0(\mathcal{S})$ may sometimes be empty. However, since N is a finite set, Π is a finite set and hence by Theorem (2.2) we have $K_1(\mathcal{S}) \neq \phi$. $K_0(\mathcal{S})$ and $K_1(\mathcal{S})$ can be considered as the predictions of our model.

5. Representation by Digraphs

Since the number of coalition structures is finite, we can represent the abstract game $(\Pi, \mathrm{dom}\,(\mathbf{g}))$ by means of a directed graph (or digraph). Let D be a digraph whose vertex set $V(D) = \Pi$ and whose arc set $A(D)$ is given by

$$A(D) = \{(\rho_1, \rho_2) \in \Pi \times \Pi : \rho_2 \text{ dom } (\mathbf{g})\ \rho_1\}.$$

We call such a digraph D the transition digraph of the abstract game $(\Pi, \mathrm{dom}(\mathbf{g}))$.

Let $(\rho_1, \rho_2) \in A(D)$. Then we say ρ_1 is adjacent to ρ_2 and ρ_2 is adjacent from ρ_1. The outdegree, $od(\rho)$, for $\rho \in \Pi$ is the number of c.s.'s adjacent from it and the indegree, $id(\rho)$, for $\rho \in \Pi$ is the number adjacent to it. Then in terms of this terminology, the core of the abstract game $(\Pi, \mathrm{dom}(\mathbf{g}))$ is given by

$$K_0(\mathbf{g}) = \{\rho \in \Pi : od(\rho) = 0\}.$$

To define the dynamic solution in terms of the transition digraph, we need a few more basic definitions from graph theory (cf. Harary [16]). A (directed) walk in a digraph is an alternating sequence of vertices and arcs $\rho_0, e_1, \rho_1, \ldots, e_n, \rho_n$ in which each arc e_i is (ρ_{i-1}, ρ_i). A closed walk has the same first and last vertex. A path is a walk in which all vertices are distinct; a cycle is a nontrivial closed walk with all vertices distinct (except the first and the last). If there is a path from ρ_1 to ρ_2, then ρ_2 is said to be accessible from ρ_1. A digraph is strongly connected or strong if any two vertices are mutually accessible. A strong component of a digraph is a maximal strong subgraph. Let $T_1, T_2, \ldots, T_m$ be the strong components of D. The condensation D^* of D has the strong components of D as its vertices, with an arc from T_i to T_j whenever there is at least one arc in D from a vertex of T_i to a vertex of T_j. (See Figure 5.1). It follows from the maximality of

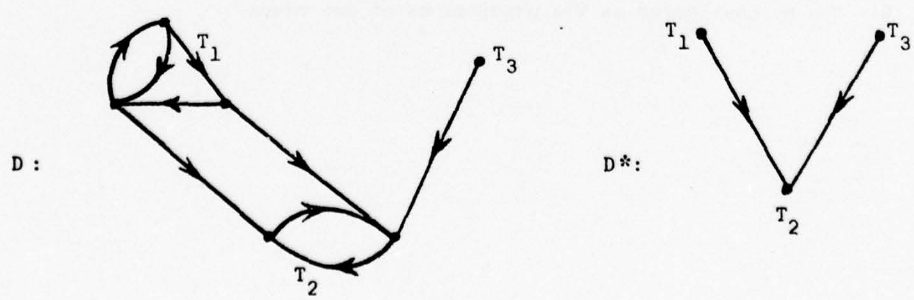

Figure 5.1 A digraph and its condensation.

of strong components that the condensation D^* of any graph has no cycles. The dynamic solution of the abstract game $(\Pi, \mathrm{dom}(\underset{\sim}{g}))$ is given by

$$K_1(\underset{\sim}{g}) = \cup\{T_i : \mathrm{od}(T_i) = 0 \text{ in the condensation } D^*\}.$$

6. A Mathematical Analysis of Caplow's Theory of Coalitions in the Triad

Much of the recent research on coalition formation in sociology and psychology was generated by a paper by Caplow [7]. Caplow proposes that the formation of coalitions

> "depends upon the initial distribution of power, and other things being equal, may be predicted under certain assumptions when the initial distribution of power is known." [7]

Caplow's four assumptions are:

A.1. Members of a triad may differ in strength. A stronger member can control a weaker member and will seek to do so.

A.2. Each member of the triad seeks control over the others. Control over two others is preferred to control over one other. Control over one other is preferred to control over none.

A.3. Strength is additive. The strength of a coalition is equal to the sum of the strengths of its two members.

A.4. The formation of coalitions takes place in an existing triadic situation, so that there is a pre-coalition condition in every triad. Any attempt by a stronger member to coerce a weaker member into joining a non-advantageous coalition will provoke the formation of an advantageous coalition to oppose the coercion.

Caplow enumerates six different triadic power structures and, based on his assumptions, makes predictions as to which coalitions will form in each type of triad. In a subsequent paper, Caplow [8] lists two more types of triads that were overlooked in the original presentation along with his predictions. The predictions are listed in Table 6.1. Before we compare our theories with Caplow's theory, we will restate Caplow's theory in a mathematical setting[†].

Let Γ be an n-person weighted majority game

$$[q;a_1,\ldots,a_n] \quad \text{where} \quad q > (a_1 + \ldots + a_n)/2, \tag{4.1}$$

and let $\mathcal{W}$ denote the set of all winning coalitions in Γ. Let i and j be two distinct players. We say that player i controls player j in coalition structure ρ iff either

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[†] The author assumes full responsibility for the ensuing formulation, which, though never formally stated, is implicit in Caplow's paper [7].

Triad Type	Distribution of Resources	Equivalent Weighted Majority Representation	Predictions: Caplow	Predictions: $K_0(\kappa)$
1	A = B = C	[2; 1,1,1]	(AB)(C), (AC)(B), (A)(BC)	(AB)(C), (AC)(B), (A)(BC)
2	A > B, B = C, A < (B+C)	[4; 3,2,2]	(A)(BC)	(A)(BC)
3	A < B, B = C	[4; 1,2,2]	(AB)(C), (AC)(B)	(AB)(C), (AC)(B)
4	A > (B+C), B = C	[3; 3,1,1]	(A)(B)(C)	(A)(B)(C), (A)(BC), (ABC)
5	A > B > C, A < (B+C)	[5; 4,3,2]	(AC)(B), (A)(BC)	(AC)(B), (A)(BC)
6	A > B > C, A > (B+C)	[4; 4,2,1]	(A)(B)(C)	(A)(B)(C), (A)(BC)
7	A > B > C, A = (B+C)	[4; 3,2,1]	(AB)(C), (AC)(B)	(AB)(C), (AC)(B), (ABC)
8	A = (B+C), B = C	[3; 2,1,1]	(AB)(C), (AC)(B)	(AB)(C), (AC)(B), (ABC)

Table 6.1

A comparison of Caplow's predictions with $K_0(\kappa)$.

(4.2) $\qquad a_i > a_j$, and $i,j \in P_k \in \mathcal{H}$, $P_k \in \rho$, or

(4.3) $\qquad i \in P_k \in \mathcal{H}$, $j \notin P_k$, $P_k \in \rho$.

Let $\beta(\rho)(i)$ denote the number of players player i controls in c.s. ρ. The <u>Caplow Power Index</u>, denoted by κ, is defined as follows:

$$\kappa(\rho)(i) = \begin{cases} \beta(\rho)(i) / \sum_{j \in N} \beta(\rho)(j) & \text{if } \sum_{j \in N} \beta(\rho)(j) \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.4}$$

for all $i \in N$ and all $\rho \in N$.

Intuitively, $\kappa(\rho)(i)$ denotes the relative power of player i when the players are aligned as in c.s. ρ †.

We are now in a position to compare Caplow's predictions with the predictions of our theory. Examples 6.1-6.8 deal with the eight different types of triads analyzed by Caplow. At the end of each example, we quote Caplow's analysis of the triad, partly to justify our definition of the Caplow power index.

<u>Example 6.1.</u> Consider the Type 1 triad $[2; \underset{A}{1}, \underset{B}{1}, \underset{C}{1}]$. Then the Caplow power index, κ, is given by

$$\kappa(\rho) = \begin{cases} (0,\ 0,\ 0) & \text{if } \rho = (A)(B)(C) \\ (1/2,\ 1/2,\ 0) & \text{if } \rho = (AB)(C) \\ (1/2,\ 0,\ 1/2) & \text{if } \rho = (AC)(B) \\ (0,\ 1/2,\ 1/2) & \text{if } \rho = (A)(BC) \\ (0,\ 0,\ 0) & \text{if } \rho = (ABC) \end{cases}$$

The transition digraph is as in Figure 6.1. $K_0(\kappa) = \{(AB)(C), (AC)(B), (A)(BC)\}$.

Caplow argues:

> "...each member strives to enter a coalition within which he is equal to his ally and stronger (by virtue of the coalition) than the isolate." [7]

<u>Example 6.2.</u> Consider the Type 2 triad $[5; \underset{A}{3}, \underset{B}{2}, \underset{C}{2}]$. Then the Caplow power index, κ, is given by

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† Note that, although Caplow stated his theory only for the restricted case of triads, our formulation of Caplow's theory holds for the more general case of n-person proper weighted majority games.

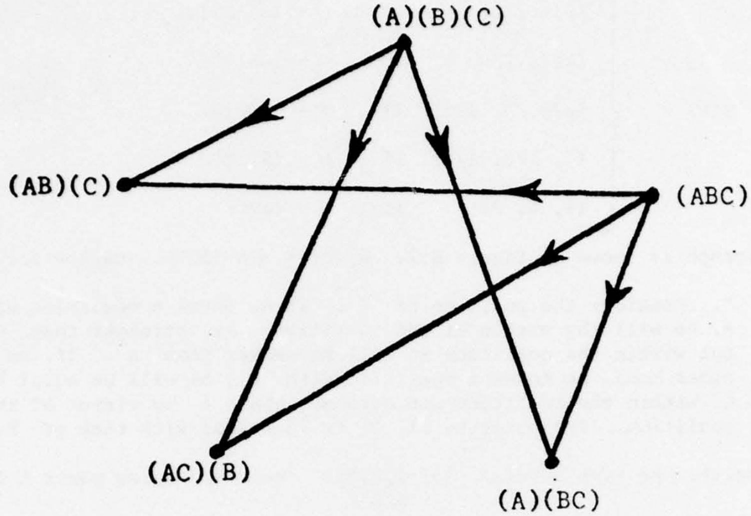

Figure 6.1. The transition digraph of Type 1 triad.

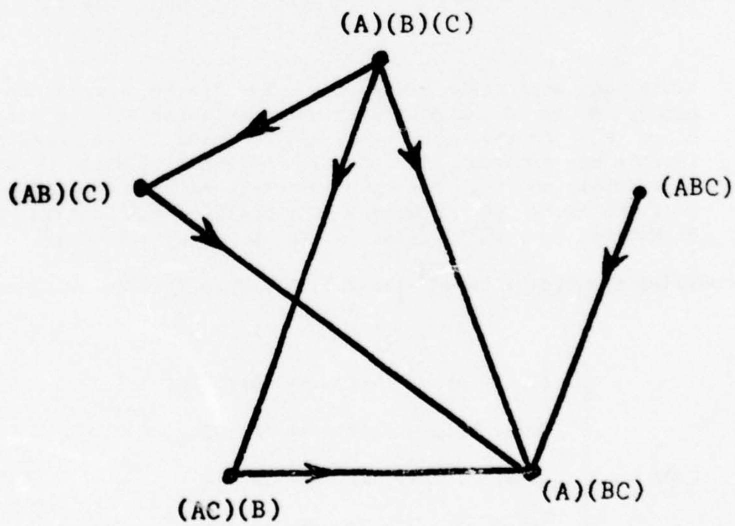

Figure 6.2. The transition digraph of Type 2 triad.

$$\kappa(\rho) = \begin{cases} (0,\ 0,\ 0) & \text{if } \rho = (A)(B)(C) \\ (2/3,\ 1/3,\ 0) & \text{if } \rho = (AB)(C) \\ (2/3,\ 0,\ 1/3) & \text{if } \rho = (AC)(B) \\ (0,\ 1/2,\ 1/2) & \text{if } \rho = (A)(BC) \\ (1,\ 0,\ 0) & \text{if } \rho = (ABC) \end{cases}$$

The transition digraph is shown in Figure 6.2. $K_0(\kappa) = \{(A)(BC)\}$. Caplow argues:

> "...Consider the position of B . If he forms a coalition with A, he will (by virtue of the coalition) be stronger than C, but within the coalition he will be weaker than A . If, on the other hand, he forms a coalition with C , he will be equal to C within the coalition and stronger than A by virtue of the coalition. The position of C is identical with that of B." [7]

Example 6.3. Consider the Type 3 triad $[3;\ \underset{A}{1},\underset{B}{2},\underset{C}{2}]$. Then the Caplow power index, κ, is given by

$$\kappa(\rho) = \begin{cases} (0,\ 0,\ 0) & \text{if } \rho = (A)(B)(C) \\ (1/3,\ 2/3,\ 0) & \text{if } \rho = (AB)(C) \\ (1/3,\ 0,\ 2/3) & \text{if } \rho = (AC)(B) \\ (0,\ 1/2,\ 1/2) & \text{if } \rho = (A)(BC) \\ (0,\ 1/2,\ 1/2) & \text{if } \rho = (ABC) \end{cases}$$

The transition digraph is shown in Figure 6.3. $K_0(\kappa) = \{(AB)(C),\ (AC)(B)\}$. Caplow argues:

> "...A may strengthen his position by forming a coalition with either B or C , and will be welcomed as an ally by either B or C . On the other hand, if B joins C , he does not improve his pre-coalition position of equality with C and superiority to A . His only motive to enter a coalition with C is to block AC. However, C's position is identical with B and he, too, will prefer A to B as an ally." [7]

Example 6.4. Consider the Type 4 triad $[3;\ \underset{A}{3},\underset{B}{1},\underset{C}{1}]$. Then the Caplow power index, κ, is given by

$$\kappa(\rho) = \begin{cases} (1,\ 0,\ 0) & \text{if } \rho = (A)(B)(C) \\ (2/3,\ 1/3,\ 0) & \text{if } \rho = (AB)(C) \\ (2/3,\ 0,\ 1/3) & \text{if } \rho = (AC)(B) \\ (1,\ 0,\ 0) & \text{if } \rho = (A)(BC) \\ (1,\ 0,\ 0) & \text{if } \rho = (ABC) \end{cases}$$

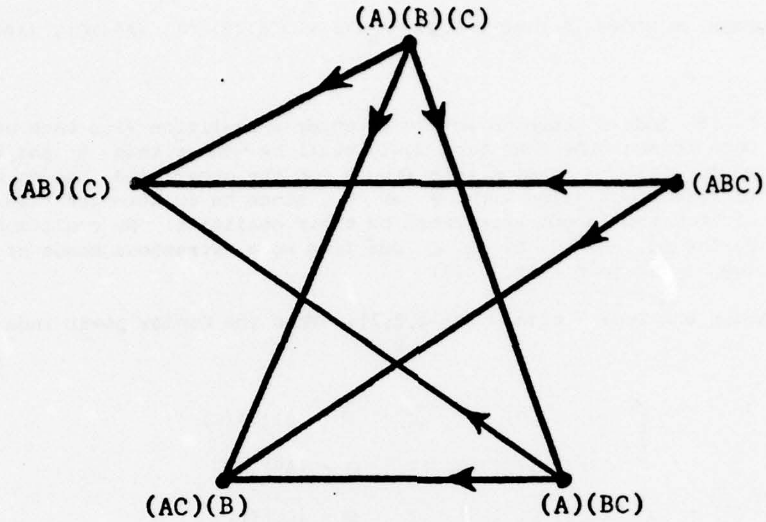

Figure 6.3. The transition digraph of Type 3 triad.

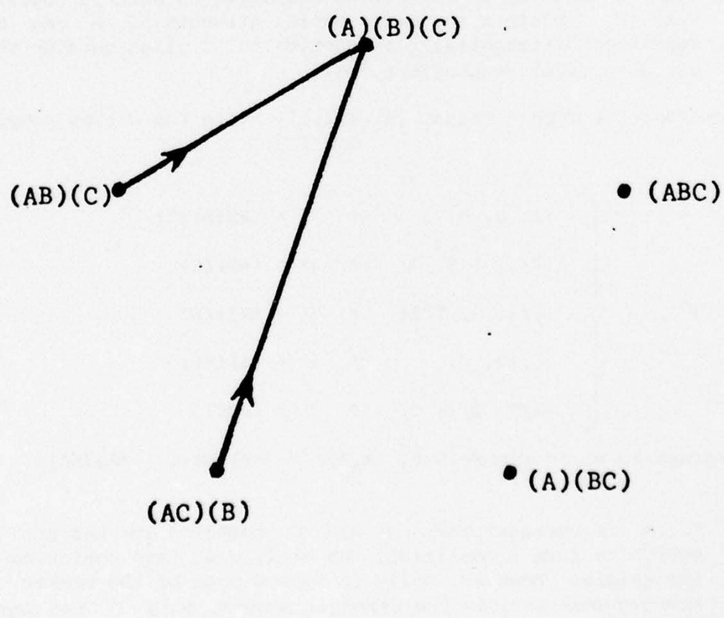

Figure 6.4. The transition digraph of Type 4 triad.

The transition digraph is shown in Figure 6.4. $K_0(\kappa) = \{(A)(B)(C), (A)(BC), (ABC)\}$.

Caplow argues:

> "...B and C have no motive to enter a coalition with each other. Once formed, the coalition would still be weaker than A and they would still be equal within it. A on the other hand, has no motive to form a coalition with B or C , since he is stronger than each of them and is not threatened by their coalition. No coalition will be formed, unless B or C can find some extraneous means of inducing A to join them." [7]

Example 6.5. Consider the Type 5 triad $[5;\ \underset{A}{4},\underset{B}{3},\underset{C}{2}]$. Then the Caplow power index, κ, is given by

$$\kappa(\rho) = \begin{cases} (0,\ 0,\ 0) & \text{if } \rho = (A)(B)(C) \\ (2/3,\ 1/3,\ 0) & \text{if } \rho = (AB)(C) \\ (2/3,\ 0,\ 1/3) & \text{if } \rho = (AC)(B) \\ (0,\ 2/3,\ 1/3) & \text{if } \rho = (A)(BC) \\ (2/3,\ 1/3,\ 0) & \text{if } \rho = (ABC) \end{cases}$$

The transition digraph is shown in Figure 6.5. $K_0(\kappa) = \{(AC)(B), (A)(BC)\}$. Caplow argues:

> "...A seeks to join both B and C and C seeks to join both A and B but B has no incentive to enter a coalition with A and has a very strong incentive to enter a coalition with C . Whether the differential strength of A and B will make them differentially attractive to C lies outside the scope of our present assumptions." [7]

Example 6.6. Consider the Type 6 triad $[4;\ \underset{A}{4},\underset{B}{2},\underset{C}{1}]$. Then the Caplow power index, κ, is given by

$$\kappa(\rho) = \begin{cases} (1,\ 0,\ 0) & \text{if } \rho = (A)(B)(C) \\ (2/3,\ 1/3,\ 0) & \text{if } \rho = (AB)(C) \\ (2/3,\ 0,\ 1/3) & \text{if } \rho = (AC)(B) \\ (1,\ 0,\ 0) & \text{if } \rho = (A)(BC) \\ (2/3,\ 1/3,\ 0) & \text{if } \rho = (ABC) \end{cases}$$

The transition digraph is as in Figure 6.6. $K_0(\kappa) = \{(A)(B)(C), (A)(BC)\}$. Caplow argues:

> "...A is stronger than B and C combined and has no motive to form a coalition. As in Type 4, true coalition is impossible. However, while in Type 4 both of the weaker members seek to join the stronger member, only C can improve his position by finding some extraneous means of inducing A to join him." [7]

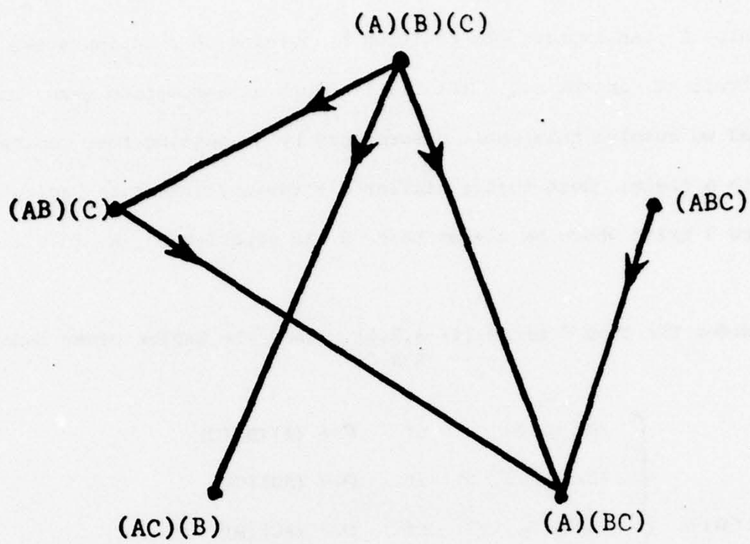

Figure 6.5 The transition digraph of Type 5 triad.

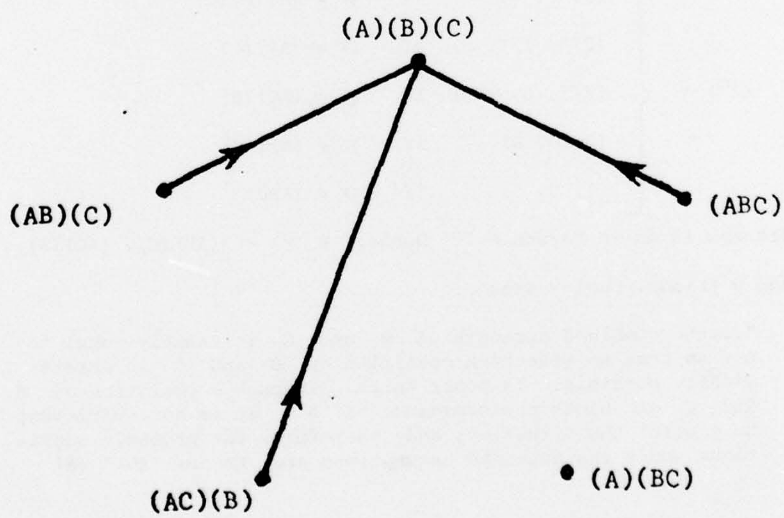

Figure 6.6 The transition digraph of Type 6 triad.

By claiming that only C can improve his position by joining A , Caplow seems to imply that B controls C in the c.s. (A)(B)(C). Such an assumption seems unreasonable to us and we resolve this small discrepancy by suggesting that Caplow has erred in making such a claim. Note that a similar discrepancy arises in Caplow's analysis of the Type 3 triad where he claims that B is superior to A in c.s. (A)(B)(C).

<u>Example 6.7.</u> Consider the Type 7 triad $[4;\ \underset{A}{3},\underset{B}{2},\underset{C}{1}]$. Then the Caplow power index, κ, is given by

$$\kappa(\rho) = \begin{cases} (0,\ 0,\ 0) & \text{if} \quad \rho = (A)(B)(C) \\ (2/3,\ 1/3,\ 0) & \text{if} \quad \rho = (AB)(C) \\ (2/3,\ 0,\ 1/3) & \text{if} \quad \rho = (AC)(B) \\ (0,\ 0,\ 0) & \text{if} \quad \rho = (A)(BC) \\ (2/3,\ 1/3,\ 0) & \text{if} \quad \rho = (ABC) \end{cases}$$

The transition digraph is shown in Figure 6.7. Hence, $K_0(\kappa) = \{(AB)(C),\ (AC)(B),\ (ABC)\}$.

<u>Example 6.8.</u> Consider the Type 8 triad $[3;\ \underset{A}{2},\underset{B}{1},\underset{C}{1}]$. Then the Caplow power index, κ , is given by

$$\kappa(\rho) = \begin{cases} (0,\ 0,\ 0) & \text{if} \quad \rho = (A)(B)(C) \\ (2/3,\ 1/3,\ 0) & \text{if} \quad \rho = (AB)(C) \\ (2/3,\ 0,\ 1/3) & \text{if} \quad \rho = (AC)(B) \\ (0,\ 0,\ 0) & \text{if} \quad \rho = (A)(BC) \\ (1,\ 0,\ 0) & \text{if} \quad \rho = (ABC) \end{cases}$$

The transition digraph is as in Figure 6.7. Hence, $K_0(\kappa) = \{(AB)(C),\ (AC)(B),\ (ABC)\}$.

For the Type 7 and 8 triads, Caplow argues:

> "...the combined strength of B and C is exactly equal to A , so that no effective coalition of B and C is strategically possible. In other words, although a coalition of B and C can block the dominance of A , it is not sufficient to control the situation, and, therefore, the probable coalitions under the standard assumptions are AB or AC." [8]

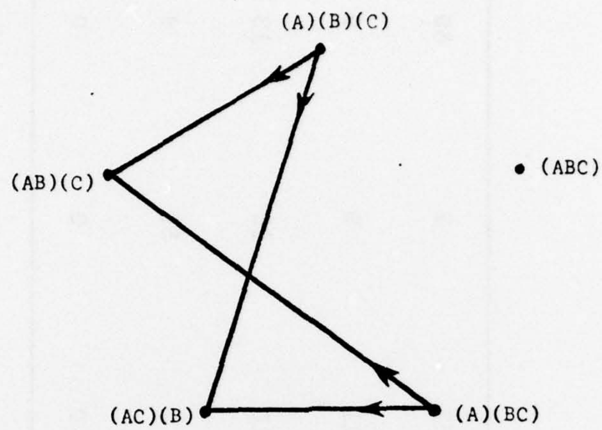

Figure 6.7. The transition digraph of Types 7 and 8 triads.

This completes our analysis of the eight different triads. The results are summarized in Table 6.1. A comparison reveals almost total agreement. All the c.s.'s predicted by Caplow are predicted by our theory. The only disagreements are in Types 4, 6, 7, 8, where our theory predicts more c.s.'s than that predicted by Caplow. However, this can easily be explained. Caplow implicitly assumes that in every triad, bargaining for coalitions start from the c.s. (A)(B)(C). A quick look at Figures 6.1-6.7 will reveal that with this additional assumption, our theory gives exactly the same predictions as Caplow's.

Vinacke and Arkoff [31] conducted experiments to test Caplow's theory. Their results, shown in Table 6.2, tend to support Caplow's theory in general with a few disagreements especially in the case of Type 3 and Type 5 triads. In the Type 3 triad, Caplow predicts coalition structures (AB)(C) and (AC)(B) without any reference to their relative frequency of occurrence. However Vinacke and Arkoff note that in the Type 3 triad, c.s. (AC)(B) occurs more frequently than c.s. (AB)(C). In the Type 5 triad, Caplow predicts coalition structures (AC)(B) and (A)(BC) with the reservation that

Type	1	2	3	4	5	6
Equivalent Weighted Majority Representation / Coalition Structures	[2; 1,1,1]	[4; 3,2,2]	[3; 1,2,2]	[3; 3,1,1]	[5; 4,3,2]	[4; 4,2,1]
(A)(B)(C)	8	1	11	62	2	60
(AB)(C)	33	13	24	11	9	9
(AC)(B)	17	12	40	10	20	13
(A)(BC)	30	64	15	7	59	8
(ABC)	2	0	0	0	0	0
Total	90	90	90	90	90	90

Table 6.2

Coalition structures formed in the six types of triads in the Vinacke-Arkoff experiments.

"...whether the differential strength of A and B will make them differentially attractive to C lies outside the scope of our present assumptions." [7]

The results of the Vinacke-Arkoff experiments indicate that in the Type 5 triad, c.s. (A)(BC) occurs more often than c.s. (AC)(B).

Chertkoff [10] makes an additional assumption which leads to the conclusion that in the Type 5 triad, c.s. (A)(BC) occurs twice as frequently as (AC)(B) and that c.s. (AB)(C) does not occur at all. Also, the same assumption when applied to the case of Type 3 triad leads to the conclusion that c.s.'s (AB)(C) and (AC)(B) are equally likely and c.s. (A)(BC) does not occur at all.

Let us assume that all transitions from each coalition structure are equally likely. Then given an initial probability distribution on the set of all coalition structures, we can compute the probability of formation of each coalition structure in $K_1(\mathfrak{g})$. E.g., in the Type 5 triad, given that players start (with probability 1) from c.s. (A)(B)(C), we observe that (Figure 6.8) c.s. (AB)(C) forms with probability 1/3, c.s. (AC)(B) forms with probability 1/3 and c.s. (A)(BC) forms with probability 1/3. However, once c.s. (AB)(C) is formed, c.s. (A)(BC) occurs with probability 1. The net result is that c.s (A)(BC) occurs with probability 2/3 and c.s. (AC)(B) occurs with probability 1/3. Coalition structure (AB)(C) also forms with probability 1/3 but only as an intermediate c.s., i.e., only temporarily.

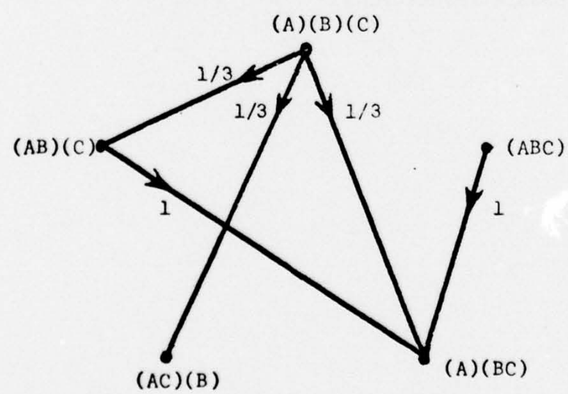

Figure 6.8 The transition digraph of the Type 5 triad with the probability of transitions under the assumption of equiprobable transitions.

A similar analysis of the Type 3 triad (Figure 6.9) indicates that, starting from c.s. (A)(B)(C) (with probability 1), c.s. (AB)(C) occurs with probability 1/2 and c.s. (AC)(B) occurs with probability 1/2. Coalition structure (A)(BC) occurs only as an intermediate coalition structure with probability 1/3. A summary of the predictions of our theories under the assumption of equi-probable transitions is shown in Table 6.3. Note that these predictions agree quite well with the Vinacke-Arkoff experimental results.

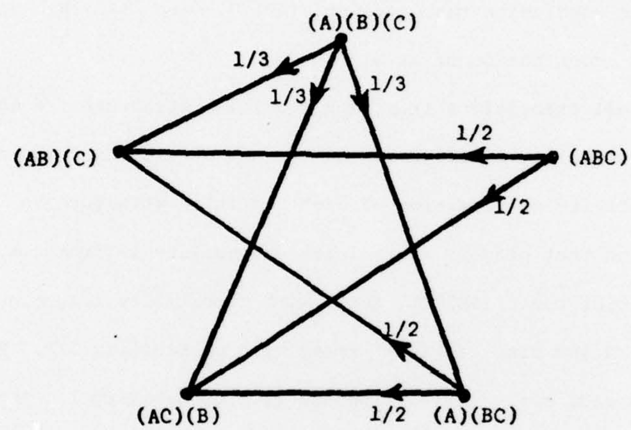

Figure 6.9. The transition digraph of the Type 3 triad with the probabilities of transition under the assumption of equi-probable transitions.

Triad Type	Equivalent Weighted Majority Representation	Starting Coalition Structure (assumed)	Probability	Intermediate Coalition Structures	Probability	Final Coalition Structures $K_1(\kappa)$	Probability
1	[2; 1,1,1]	(A)(B)(C)	1			(AB)(C) (AC)(B) (A)(BC)	1/3 1/3 1/3
2	[4; 3,2,2]	(A)(B)(C)	1	(AB)(C) (AC)(B)	1/3 1/3	(A)(BC)	1
3	[3; 1,2,2]	(A)(B)(C)	1	(A)(BC)	1/3	(AB)(C) (AC)(B)	1/2 1/2
4	[3; 3,1,1]	(A)(B)(C)	1			(A)(B)(C)	1
5	[5; 4,3,2]	(A)(B)(C)	1	(AB)(C)	1/3	(AC)(B) (A)(BC)	1/3 2/3
6	[4; 4,2,1]	(A)(B)(C)	1			(A)(B)(C)	1
7	[4; 3,2,1]	(A)(B)(C)	1			(AB)(C) (AC)(B)	1/2 1/2
8	[3; 2,1,1]	(A)(B)(C)	1			(AB)(C) (AC)(B)	1/2 1/2

Table 6.3.

A summary of the predictions of the c.s. model under the assumption of equi-probable transitions.

7. A Mathematical Analysis of Gramson's Theory of Coalition Formation

Following Caplow, Gamson formulated a slightly more general theory of coalition formation in proper weighted majority games without dictators or veto players. Before we present Gamson's theory, we need a definition. Let Γ be a weighted majority game. A cheapest winning coalition is a winning coalition whose total weight is a minimum among all winning coalitions. Gamson's main hypothesis is as follows:

> "Any participant will expect others to demand from a coalition a share of the payoff proportional to the amount of resources which they contribute to a coalition." [13]

Here, a participant refers to a player, and his resources refers to his weight in the weighted majority game. Based on his main hypothesis, Gamson makes the following predictions about coalition formation.

(i) A player will favor a cheapest winning coalition.

(ii) A coalition of two distinct players $\{i,j\}$ will form if and only if there are reciprocal strategy choices between the two players. I.e. both player i and player j prefer coalition $\{i,j\}$.

(iii) The process of coalition formation is a step by step process where two players merge together into a coalition at a time.

(iv) Once a two-person coalition forms, the situation becomes a new one--the two players in the coalition are replaced by one player whose weight equals the sum of the weights of the two players in the coalition.

Implicit in Gamson's main hypothesis is a definition of a power index. Let $\Gamma = [q; a_1,\dots,a_n]$ be a proper weighted majority game without a dictator or veto players. Then the Gamson power index, denoted by γ, is given by

$$\gamma(\rho)(i) = \begin{cases} \dfrac{a_i}{\sum_{i \in P_k} a_i} & \text{if } \sum_{i \in P_k} a_i \neq 0 \text{ and } P_k \in \mathcal{W} \\ 0 & \text{if } \sum_{i \in P_k} a_i = 0 \text{ or } P_k \notin \mathcal{W} \end{cases} \tag{7.1}$$

where $P_k \in \rho$ is such that $i \in P_k$, for all $\rho \in \Pi$ and all $i \in N$. Let

$$g = \min_{R \in \mathcal{W}} \sum_{i \in R} a_i \tag{7.2}$$

and

(7.3) $\Pi_g = \{\mathcal{P} \in \Pi : \mathcal{P}$ contains a cheapest winning coalition$\}$.

Then Theorem 7.1 tells us what our model predicts.

Theorem 7.1 Let Γ be a proper weighted majority game. Then $K_0(\gamma) = \Pi_g$.

Proof: Let $\mathcal{P}_1 \in \Pi_g$. Suppose $\mathcal{P}_2 \in \Pi$ such that $\mathcal{P}_2 \text{ dom}_R(\gamma) \mathcal{P}_1$ for some $R \in \mathcal{P}_2$ with $R \in \mathcal{W}$. Then $\gamma(\mathcal{P}_2)(i) > \gamma(\mathcal{P}_1)(i)$ for all $i \in R$. Let $T \in \mathcal{P}_1$ such that $T \in \mathcal{W}$ and $\sum_{i\in T} a_i = g$. Since Γ is proper, $R \cap T \neq \emptyset$. Let $j \in R \cap T$. Then $\gamma(\mathcal{P}_1)(j) = a_j/g$. Since $j \in R$, $\gamma(\mathcal{P}_2)(j) = a_j/(\sum_{i\in R} a_i) > a_j/g$; i.e., $\sum_{i\in R} a_i < g$ and a contradiction (from the definition of g) results. Hence $K_0(\gamma) \supset \Pi_g$.

Let $\mathcal{P}_1 \in \Pi_g$ and $\mathcal{P}_2 \in \Pi$ such that $\mathcal{P}_2 \notin \Pi_g$. Then $\mathcal{P}_1 \text{ dom}_T(\gamma) \mathcal{P}_2$ where $T \in \mathcal{P}_1$ such that $T \in \mathcal{W}$ and $\sum_{i\in T} a_i = g$, because $\gamma(\mathcal{P}_1)(i) = a_i/g$ for all $i \in T$ and $\gamma(\mathcal{P}_2)(i) < a_i/g$ for all $i \in T$. Hence $K_0(\gamma) \subset \Pi_g$. □

It can be easily shown that Gamson's predictions (i)-(iv) about coalition formation lead to c.s.'s in Π_g. However Gamson assumes that players begin forming coalitions starting from one player coalitions. So if we choose only those c.s.'s in Π_g that are accessible from the c.s. consisting of only one player coalitions, our model reaches the same conclusions as Gamson's predictions.

8. Conclusion

Under the same assumptions, our theory of coalition formation makes the same predictions as Caplow's and Gamson's theories. This, however, should not be misinterpreted as an endorsement of these two theories. Both Caplow's and Gamson's theories are descriptive and depend heavily on their (implicit) definition of a power index. From a normative point of view these power indices have many shortcomings. Several power indices have been defined for simple games. Two of these, the Shapley-Shubik index [25] and the Banzhaf-Coleman index [2, 3, 4, 5, 11] have been extensively used and studied. Hence it is most appropriate to study the predictions of our model with respect to these power indices. A detailed analysis of the predictions of our theory with respect to the Shapley-Shubik power index is presented in Shenoy [28].

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In this paper, ~~we~~ proposes a theory of coalition formation in simple games. The process of coalition formation is modeled as an abstract game. Two solutions of abstract games, the core and the dynamic solution, are used as the predictions of our model. Two classical theories of coalitions in sociology due to Caplow and Gamson are reformulated in a more general and mathematical setting. These theories are then analyzed using the techniques of our theory.

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